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Relaxation phenomena in two interacting spin lattices

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Abstract. We study the kinetics of the formation of antiferromagnetic order in a finite spin system. The system is initially forced to be an unstable ferromagnetic state by means of an external field. When the field is switched off the system decays toward equilibrium. The finite size of the system has relevant consequences on the time of persistence of the initial unstable state. The fluctuations driving the system to equilibrium become anomalously large in the transient. We obtain analytic results for the transient fluctuation statistics in the limit where the average squared magnetisation relaxes much more quickly than relative orientation of the sublattices. The results have been obtained in the framework of spherical model and of molecular field approximation. The relaxation dynamics are ruled by the non-linear Langevin equations.

1. Introduction

In this paper we address the phenomenon of the recovering of equilibrium long range order in a Néel lattice initially set in an unstable state. The choice of a Néel lattice with an infinite range interaction is equivalent, for the static properties, to the mean field approximation. The spins belonging to the same sublattice interact ferromagnetically whereas an antiferromagnetic interaction couples the two sublattices (Smart 1966). Below the Néel temperature the system exhibits antiferromagnetic order with opposite average magnetisations in the two sublattices. When an external magnetic field of sufficient strength is applied, the spins of the downward sublattice flip in the field direction, realising a ferromagnetic state whose thermodynamic properties may be derived by solving the mean field equations in the presence of the external field. As the field is switched off the ferromagnetic state becomes unstable and undergoes a relaxation towards the equilibrium Néel state.

The aim of this paper is to study the dynamics of this decay process with particular regard to the effect of the finite size of the system on the onset of the relaxation process. Finite size effects have already turned out to be quite important in the growth of ferromagnetic order after a thermal quenching (Ciuchi *et al* 1988). In particular it was found that fluctuations due to the finite size of the system, usually negligible in the

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thermodynamic limit, are amplified to a macroscopic amplitude in the decay from an unstable state. However, whereas in a ferromagnet this phenomenon is completely obscured by the presence of an external field that is small but finite in the thermodynamic limit, in the present case it occurs for a finite magnitude range of the applied field.

An analytic theory of the phenomenon is easily developed in the framework of the spherical model (Berlin and Kac 1952) which is equivalent to the weak constraint (Lyons and Kaplan 1960) used for the determination of the ground state spin configuration in the steady limit. In this context the site magnetisation—the natural variable to describe the time evolution of the system—is assumed to be a stochastic process obeying a generalised Langevin equation. We will assume Langevin dynamics rule both the steady state and the non-linear relaxation. To be more precise, we shall identify the time-dependent local magnetisation with a $2N$-component scalar field defined on each lattice site. The time evolution of the local magnetisation, after switching off of the field, is due to the action of three effects: the coupling with the magnetisation at different sites, treated in the molecular field approximation; the random local thermal noise; and the presence of non-linear terms that force the system to relax according to the weak constraint. The main advantage of this model is the possibility of describing the system with $2N$ degrees of freedom in terms of a statistically equivalent one with only two degrees of freedom, namely the two average sublattice magnetisations. These evolve according to non-linear Langevin equations where the noise terms are vanishingly small in the thermodynamic limit. An interesting feature of the present model is that the two time-dependent variables can be chosen *ad hoc*, according to the actual property to be investigated. Thus the representation of the degree of freedom as the 'modulus' and 'phase' of the two-component magnetisation vector, defined from the sublattice average magnetisations, is particularly useful in order to point out the mechanism underlying the transition from ferromagnetic to antiferromagnetic order in the presence of a magnetic field. During the decay the phase relaxation is decoupled from that of the modulus and finite size fluctuations are relevant only on the phase. Moreover we individuate an adiabatic regime, where the 'modulus' of the magnetisation vector attains its steady state value very quickly before the orientation relaxation starts towards the final antiferromagnetic configuration. This orientation relaxation process is easily shown to be an example of a decay process from an initial unstable state (Suzuki 1980, de Pasquale and Tombesi 1979, de Pasquale *et al* 1982). A simple analytic theory for characterising the relaxation properties is then applied in terms of total and staggered magnetisation. For these quantities and for their probability distributions the analytic results are compared with those from numerical solutions. We also consider time averages over finite time intervals, obtaining results that show the qualitative behaviour expected for the decay of an unstable state (de Pasquale *et al* 1982).

2. The model

Our system consists of $2N$ spins distributed on two sublattices labelled A and B, each one with N sites. We introduce coupling parameters for intralattice and interlattice spin–spin interactions denoted by J_F and J_{AF}, respectively. Denoting by $\psi_i(t)$ and $\varphi_i(t)$ the spin variables for the sublattice A and B respectively, the non-linear Langevin equations can be written as follows:

$$\mathrm{d}\varphi_i(t) = [J_F m_A - J_{AF} m_B + (r_0 - uA)\varphi_i(t)]\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}\omega_i(t) + h(t)\,\mathrm{d}t \qquad (1a)$$

$$\mathrm{d}\psi_i(t) = [J_{AF}m_A - J_F m_B + (r_0 - uA)\psi_i(t)]\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}\omega_i'(t) + h(t)\,\mathrm{d}t \tag{1b}$$

where $h(t)$ is a homogeneous external field existing only for times $t < 0$. The first two linear terms in (1) take into account the interactions among sites i ($i = 1, N$) in the molecular field approximation whereas the third and fourth terms are the stochastic increments due to the thermal bath. The terms ω_i, ω_i' are independent Wiener processes for which the usual properties hold:

$$\langle \mathrm{d}\omega_i(t)\,\mathrm{d}\omega_j(t')\rangle = \delta_{ij}\delta(t-t')\,\mathrm{d}t \qquad \langle \mathrm{d}\omega_i'(t)\,\mathrm{d}\omega_j'(t')\rangle = \delta_{ij}\delta(t-t')\,\mathrm{d}t$$

$$\langle \mathrm{d}\omega_i(t)\,\mathrm{d}\omega_j'(t')\rangle = 0 \qquad \langle \mathrm{d}\omega_i(t)\rangle = \langle \mathrm{d}\omega_i'(t)\rangle = 0.$$

In (1) the lattice average of the squared field A is given by

$$A = \frac{1}{2N}\left(\sum_i \varphi_i(t)^2 + \sum_i \psi_i(t)^2\right). \tag{2}$$

Finally, the magnetisation on either sublattice is defined as

$$m_A = \frac{1}{N}\sum_i \varphi_i(t) \qquad m_B = \frac{1}{N}\sum_i \psi_i(t). \tag{3}$$

The weak constraint (spherical limit) is satisfied in the limit in which the evolution of the magnetisation field occurs on a hypersphere in the phase space where A is constant. In this limit, as shown below, we obtain a statistically equivalent description of the sublattice magnetisations in terms of only two non-linear Langevin equations.

From equations (1) and (2), according to the Ito prescriptions (Gardiner 1983) and taking (3) into account, we obtain the generalised Langevin equations for the lattice average quantities:

$$\dot{A} = J_F m_A^2 + J_F m_B^2 - 2J_{AF}m_A m_B + 2(r_0 - uA)A + \frac{\sqrt{\varepsilon}}{N}\left(\sum_i \xi_i(t)\varphi_i(t) + \sum_i \xi_i'(t)\psi_i(t)\right) + 2\varepsilon + h(t)(m_A + m_B) \tag{4}$$

$$\dot{m}_A = J_F m_A - J_{AF}m_B + (r_0 - uA)m_A + \frac{\sqrt{\varepsilon}}{N}\sum_i \xi_i(t) + h(t) \tag{5a}$$

$$\dot{m}_B = -J_{AF}m_A + J_F m_B + (r_0 - uA)m_B + \frac{\sqrt{\varepsilon}}{N}\sum_i \xi_i'(t) + h(t) \tag{5b}$$

where the thermal noise terms $\xi_i(t) = \dot{\omega}_i(t)$ and $\xi_i'(t) = \dot{\omega}_i'(t)$ appear as formal derivatives of the N-dimensional Wiener processes ω_i, ω_i' and have the usual statistical properties:

$$\begin{aligned}\langle \xi_i(t)\rangle &= 0 \qquad \langle \xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t-t')\\ \langle \xi_i'(t)\rangle &= 0 \qquad \langle \xi_i'(t)\xi_j'(t')\rangle = \delta_{ij}\delta(t-t').\end{aligned} \tag{5c}$$

It is worth noting that the noise strengths in (4) and (5) are vanishingly small in the large N limit.

The actual reduction of the degrees of freedom is achieved by performing the *spherical* limit (Sherrington 1981), i.e., $r_0, u \to \infty$, $r_0/u = 1$. Writing $A = 1 + \lambda/r_0$ in (4) we obtain

$$\lambda = (J_F/2)m_A^2 + (J_F/2)m_B^2 - J_{AF}m_A m_B + \varepsilon + \tfrac{1}{2}\sqrt{\varepsilon}/N\,\Xi(t) + h(t)(m_A + m_B)/2 \tag{6a}$$

$$\dot{m}_A = J_F m_A - J_{AF}m_B - \lambda m_A + \sqrt{\varepsilon}/N\,\xi_A(t) \tag{6b}$$

$$\dot{m}_B = J_{AF} m_A - J_B m_B - \lambda m_B + \sqrt{\varepsilon/N}\, \xi_B(t) \tag{6c}$$

where

$$\Xi(t) = \left(\sum_i \varphi_i(t) \xi_i(t) + \sum_i \psi_i(t) \xi_i'(t) \right) \Big/ \sqrt{N}$$

$$\xi_A(t) = \frac{1}{\sqrt{N}} \sum_i \xi_i(t) \qquad \xi_B(t) = \frac{1}{\sqrt{N}} \sum_i \xi_i'(t).$$

We notice that, due to the spherical limit, the noise term Ξ, with unitary variance, is a Gaussian white noise with the following statistical properties

$$\langle \Xi(t) \rangle = 0 \qquad \langle \Xi(t)\Xi(t') \rangle = A(t)\delta(t-t'). \tag{7}$$

It is, however, correlated to the sublattice magnetisations by

$$\langle \Xi(t) \xi_{A,B}(t') \rangle = m_{A,B}\, \delta(t-t').$$

The above average must be understood as done over an ensemble of noise realisations with fixed $m_{A,B}$. Substituting (6*a*) into (6*b*, *c*) gives the required time evolution for the magnetisation of each sublattice

$$\dot{m}_A = (J_F - \varepsilon) m_A - J_{AF} m_B - (m_A/2)(\gamma(m_A, m_B) + \sqrt{\varepsilon/N}\, \Xi(t)) + \sqrt{\varepsilon/N}\, \xi_A(t) + h(t) \tag{8a}$$

$$\dot{m}_B = (J_F - \varepsilon) m_B - J_{AF} m_A - (m_B/2)(\gamma(m_A, m_B) + \sqrt{\varepsilon/N}\, \Xi(t)) + \sqrt{\varepsilon/N}\, \xi_B(t) + h(t) \tag{8b}$$

where

$$\gamma(m_A, m_B) = J_F m_A^2 + J_F m_B^2 - 2 J_{AF} m_A m_B + h(t)(m_A + m_B).$$

With equation (8) we have reduced the original $2N$ Langevin field equations (1) to two coupled stochastic equations for the sublattice average magnetisations, which describe the macroscopic properties of interest completely. This is one of the most relevant features of our model. Finite size effects have been taken into account exactly; they appear in the two noise sources. The next step is to study under which conditions fluctuations induced by these terms are amplified to a macroscopic size. To do that we discuss the unstable behaviour of the system in terms of the new quantity $M(t)$ as

$$M(t) = R(t)\, e^{i\theta(t)} = m(t) + i m_s(t) \tag{9}$$

where we have introduced the total and staggered magnetisation m and m_s, respectively given by

$$m(t) = (m_A(t) + m_B(t))/2 = R(t)\cos(\theta(t)) \tag{10a}$$

$$m_s(t) = (m_A(t) - m_B(t))/2 = R(t)\sin(\theta(t)). \tag{10b}$$

Equating real and imaginary parts of $\dot{M}$ one arrives, after straightforward manipulations, at the time evolution equations for the 'phase' $\theta(t)$ and the 'modulus' $R(t)$ of $M(t)$

$$\dot{\theta} = J_{AF} \sin 2\theta - (h/R)\sin\theta + (\sqrt{\varepsilon}/2N)\, \xi_\theta / R \tag{11a}$$

$$\dot{R} = R(\kappa_\theta - \varepsilon - \tfrac{1}{2}\Xi \sqrt{\varepsilon/N}) - \kappa_\theta R^3 - hR^2 \cos\theta + \xi_R \sqrt{\varepsilon/2N} \tag{11b}$$

with

$$\kappa_\theta = J_F - J_{AF}\cos 2\theta$$

and where $\xi_{R,\theta}$ are independent Gaussian noise terms with the same statistical properties as $\xi_{A,B}$.

3. Steady state properties

In order to investigate the dynamic evolution of our system between the initial and final state we need to know its static properties. These are better studied in terms of the total and staggered magnetisations just defined. From (10) and from the deterministic limit of (8) we have

$$\dot{m} = J_1 m - (J_1 m^2 + J_2 m_s^2 + hm + \varepsilon)m + h \tag{12a}$$

$$\dot{m}_s = J_2 m_s - (J_1 m^2 + J_2 m_s^2 + hm + \varepsilon)m_s \tag{12b}$$

with

$$J_1 = J_F - J_{AF} \qquad J_2 = J_F + J_{AF}.$$

We first look for the stable long range antiferromagnetic solution, i.e., $m_s \neq 0$. This is straightforwardly given by:

$$m_s = \pm\sqrt{1 - \varepsilon/J_2 - h^2/4J_{AF}^2} \tag{13}$$

$$m = h/2J_{AF}. \tag{14}$$

From these solutions one sees the existence of a critical temperature $\varepsilon_c = J_2$ determining the passage from paramagnetic to ordered phase. Here and in what follows we shall restrict to the phase where $\varepsilon < \varepsilon_c$, moreover we shall assume that $J_F > J_{AF}$. It will be clear in the next section that with these conditions the evolution of the system may be analysed in terms of only one relevant variable. Below the critical temperature the long range order depends on the applied field. From (13) we see that there exists a critical field h_c, whose magnitude determines whether the ordered phase will be ferromagnetic ($h > h_c$) or antiferromagnetic ($h < h_c$). The value of the critical field is given by

$$h_c(\varepsilon) = \pm 2J_{AF}\sqrt{1 - \varepsilon/\varepsilon_c}.$$

Therefore for $h > h_c$ the stable state is the one with $m \neq 0$ and $m_s = 0$, i.e., with the average magnetisation oriented parallel to h on both sublattices. This will be the initial state of our system whose magnetisation may be obtained from the solution of the following cubic equation:

$$J_1 m - J_1 m^3 + (J_1 - \varepsilon)m + h = 0.$$

The behaviours of m_s and m as a function of h are shown in figure 1(a).

It is now interesting to study the steady properties of the variables θ and R since it will appear that, under certain conditions, the dynamics of the system is ruled by the evolution of θ only. By inverting equation (10) the equilibrium values of R and θ for $h < h_c$ are given by

$$R = \sqrt{1 - \varepsilon/J_2} \qquad \theta = \tan^{-1}\sqrt{h_c^2/h^2 - 1} \tag{15}$$

and for $h > h_c$ by

$$R \simeq \sqrt{1 - \varepsilon/h} \qquad \theta = 0 \tag{16}$$

(R is evaluated here in the limit $h \gg h_c$). The behaviours of R and θ as a function of h

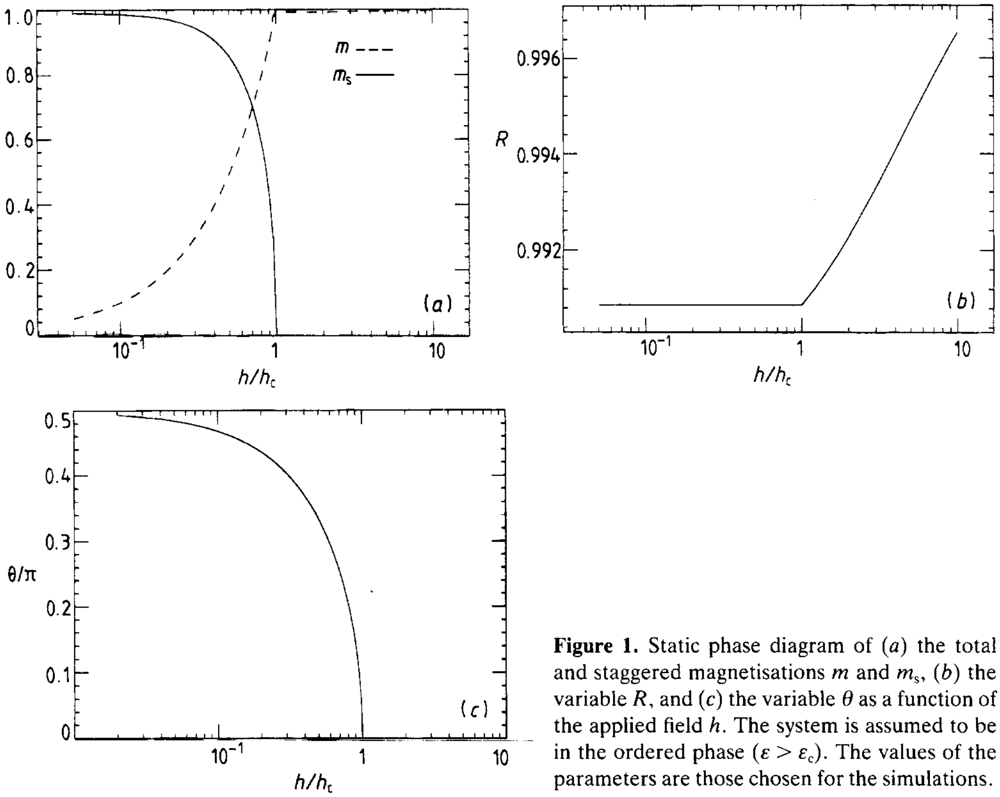

Figure 1. Static phase diagram of (*a*) the total and staggered magnetisations m and m_s, (*b*) the variable R, and (*c*) the variable θ as a function of the applied field h. The system is assumed to be in the ordered phase ($\varepsilon > \varepsilon_c$). The values of the parameters are those chosen for the simulations.

are shown in figure 1(*b*, *c*). It must be noted that the variable R does not undergo any transition at the critical field if $\varepsilon = 0$. For finite but small temperatures the difference between $R(h = 0)$ and $R(h \gg h_c)$ is of order ε.

Due to this behaviour of R, the term θ turns out to be the variable describing the instability of the system at the critical field. In fact one can write a potential function $V(\theta) = -\mathrm{d}\theta/\mathrm{d}t$, assuming R is constant:

$$V(\theta) = (J_{AF}/2)\cos 2\theta - (h/R)\cos\theta. \tag{17}$$

From equation (17) the instability of the state with $\theta = 0$ appears as soon as $h < 2J_{AF}$. In the approximations used so far, θ comes out to be the variable suitable to describe the kinetic instability of the system after the external field is switched off and we will use it in the next section.

4. Ordering kinetics

We now turn to the dynamics of the system, starting from the stable state in which $h \gg h_c$ and evolving towards a new equilibrium state when the field is instantaneously switched off at $t = 0$. We assume the temperature to be kept constant and below the critical temperature throughout the whole process. We shall first discuss the non-linear relaxation properties in the deterministic limit and then consider the role of thermal fluctuations. For $t > 0$ and for $h = 0$ we obtain from (11*a*) that the θ relaxation is decoupled from the R relaxation. The decay of $\theta(t)$ is given by

$$\theta(t) = \tan^{-1}[\tan(\theta(0)) \exp(t/\tau_\theta)] \tag{18}$$

where

$$\tau_\theta = 1/2J_{\mathrm{AF}} \tag{19}$$

is the time constant for the θ relaxation. A simple analytic solution of (11*b*) for $R(t)$ is possible in the adiabatic limit, i.e. when the relaxation processes for $R(t)$ and $\theta(t)$ occur on a very different time scale. This depends, in general, on the relative magnitudes of the time constants τ_θ and τ_R for $R(t)$. To find τ_R we solve (11*b*) for fixed θ and obtain

$$R(t) = R(0)\{1 + [\kappa_\theta/(\kappa_\theta - \varepsilon)]R^2(0)(1 - \exp(-2t/\tau_R))\}^{-1/2} \tag{20}$$

where the time constant for the $R(t)$ relaxation is

$$\tau_R = 1/|(\kappa_\theta - \varepsilon)|. \tag{21}$$

By comparing τ_θ with τ_R one sees that the adiabatic regime is realised only when the interlattice coupling constant, J_{AF}, is small with respect to the intralattice one, J_{F}. For the steady state value of (21), R^* is only slightly affected by the θ motion in the low-temperature limit

$$R^*(\theta) = \sqrt{(1 - \varepsilon/\kappa_\theta)}. \tag{22}$$

From equations (18) and (20) one sees that when $\tau_\theta \gg \tau_R$, the R relaxation takes place first and depends, through κ_θ, on $\theta(0)$. We note that the adiabatic condition corresponds to a long θ lifetime that can be affected by fluctuations once the R relaxation has occurred.

We now want to consider the effect of the thermal noise terms on the ordering kinetics. As we will show, these terms will affect the variable, θ, relaxing from an extremum of the potential $V(\theta)$, whereas they are practically ineffective on $R(t)$. According to the initial conditions of the system just described, we can write that $\theta(0) = 0$, and $R(0)$ is given by equation (16*a*).

From the complete relaxation of $R(t)$, equation (11*b*) with $h = 0$ at $t > 0$, one sees that the noise terms initially do not affect the behaviour of $R(t)$ since its initial deterministic drift is large compared to the fluctuations. Moreover, according to the deterministic evolution, R remains always much larger than the noise-induced fluctuations. Therefore the noise terms can be neglected in considering the time evolution of R, which is equivalent to taking for R the thermodynamic limit directly in the dynamic equations (11*a*, *b*). From (22) one sees that for $t > \tau_R$, R reaches the value $R^*(\theta = 0)$.

The next relaxation of θ does not affect appreciably the evolution of $R(t)$, since in the adiabatic regime these two values of R are almost equal. We can therefore concentrate our attention on (11*a*) considering $R(t) \simeq R_{\mathrm{eq}}$, with R_{eq} given in equation (15). It is possible to solve these stochastic equations in the quasi-deterministic theory (QDT)

(Suzuki 1980). We may estimate the process $\theta(t)$ using the solution (18) of the deterministic version of (11*a*) with an initial datum θ_0 that is a random variable given by

$$\theta_0(t) = \sqrt{\frac{\varepsilon}{2N}} \int_0^t \mathrm{d}t' \exp(-2J_{\mathrm{AF}}t')\xi_\theta(t')/R_{\mathrm{eq}} \tag{23}$$

with average

$$\langle \theta_0(t) \rangle = 0 \tag{24a}$$

and variance

$$\sigma^2(t) = \langle \theta_0(t)^2 \rangle - \langle \theta_0(t) \rangle^2 = \sigma^2(1 - \exp(-4J_{\mathrm{AF}}t)) \tag{24b}$$

where

$$\sigma^2 = \varepsilon/8NJ_{\mathrm{AF}}R_{\mathrm{eq}}^2. \tag{24c}$$

We can estimate the time the system spends close to the unstable ferromagnetic state when $\theta = 0$ by considering the time τ_0 in which $\theta^2(t)$ attains a value appreciably different from zero:

$$\langle \theta^2(t) \rangle \simeq \exp(4J_{\mathrm{AF}}t)\sigma^2(t). \tag{25}$$

We obtain

$$\tau_0 \simeq (1/4J_{\mathrm{AF}}) \ln(1/\sigma^2). \tag{26}$$

After a time comparable with τ_0 any realisation of the system follows a deterministic decay driven completely by the evolution of θ in the adiabatic limit. Equations (24–26) show that the finite size N of the system is essential for the θ decay from the unstable state within a finite time. The above results allow us to obtain the time behaviour of the total and staggered magnetisation, and its higher moments, as well as to evaluate the whole probability distribution function (PDF) of m_s^2. The interesting quantities are

$$\langle m^2(t) \rangle = R^2(t)\langle \cos^2\theta(t) \rangle \tag{27a}$$

$$\langle m_s^2(t) \rangle = R^2(t)\langle \sin^2\theta(t) \rangle \tag{27b}$$

and the generic even moment (the odd ones being identically zero for symmetry reasons due to the absence of external field):

$$\langle m_s^{2k}(t) \rangle = R^{2k}(t)\langle \sin^{2k}\theta(t) \rangle. \tag{28}$$

In the above equations $R(t)$ is considered a deterministic quantity. Moreover $R^{2k}(t) \to R_{\mathrm{eq}}^{2k}$ in a time $t \ll \tau_0$. From these facts we have that, in the dynamical context, the presence of unstable states that we have found dealing with the steady state properties of the system, is related to long persistences around such states. A vanishingly small thermal noise is able to drive the system into the final stable state. Nevertheless anomalous fluctuation will appear during the transient behaviour.

Analytic expressions for the average quantities in equations (27), (28) are obtained by considering the following identities

$$\sin^{2k}\theta = [\tan^2\theta/(\tan^2\theta + 1)]^k \tag{29}$$

where $\tan^2\theta$ is given by

$$\tan^2\theta(t) = (\exp(-2J_{AF})\theta_0(t))^2 = w^2/z^2(t) \tag{30}$$

in terms of the variable

$$z(t) = (1/\sigma(t))\exp(-2J_{AF}t). \tag{31}$$

In (30) we put $\theta_0 = \sigma(t)\omega$, with ω a Gaussian variable having zero average and unitary variance. Hence the thermal average reduces in QDT to an average over ω:

$$\langle\sin^{2k}\theta(t)\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}w\, \mathrm{e}^{\omega^2/2}\left(\frac{w^2}{w^2 + z^2(t)}\right)^k. \tag{32}$$

The integral in (32) can be given an analytic expression in terms of the parabolic cylinder function $D_{-2k}(x)$ (Abramowitz and Stegun 1965):

$$\langle\sin^{2k}\theta(t)\rangle = [(2k-1)!!/2]\exp(z^2/4D_{-2k}[z(t)].$$

The term D_{-2k} can be evaluated in an approximate way to find the average staggered magnetisation. In figure 2(*a*, *b*) we show the results obtained by computer simulations and compared with the analytic solutions obtained in the QDT, equations (27), (28). It is interesting to investigate the behaviour of the fluctuations of m_s^2. To do this we introduce the following time-dependent variance

$$\langle\Delta m_s^4\rangle = \langle m_s^4(t)\rangle - \langle m_s^2(t)\rangle^2. \tag{33}$$

In figure 2(*c*) we observe an anomalous behaviour of the fluctuations in the transient ($t \simeq 25$) when the staggered magnetisation begins to grow. The fluctuations in this time range do not decrease as the size of the system increases but become size independent. To correctly interpret the simulation results one has to bear in mind the following point. The finite values used for $r_0 = u = 10$ instead of their spherical limit (see equations (6)) bring about a systematic shift of the equilibrium values of all the non-vanishing calculated quantities. To take this fact into account in the comparison between simulations and theory, we have used for the equilibrium value R_{eq} the following expression instead of that in equation (15)

$$R_{eq}^2 = (r_0 + J_{AF} + J_F)/u - \varepsilon/(J_F + J_{AF}).$$

This follows from the deterministic value of (4) with $h = 0$, (5) and from the definition (10). The dynamic behaviour is, however, not affected by the value of r_0, u.

5. The ordering statistics

In a recent work (Mazenko *et al* 1988) it has been proposed to characterise a system exhibiting spontaneous growth of order with an order parameter field which splits into an ordering component plus a fluctuating field. The ordering component is affected by fluctuations in the relaxation regime and achieves a deterministic value asymptotically. In our case a natural choice for the definition of the ordering component is the QDT

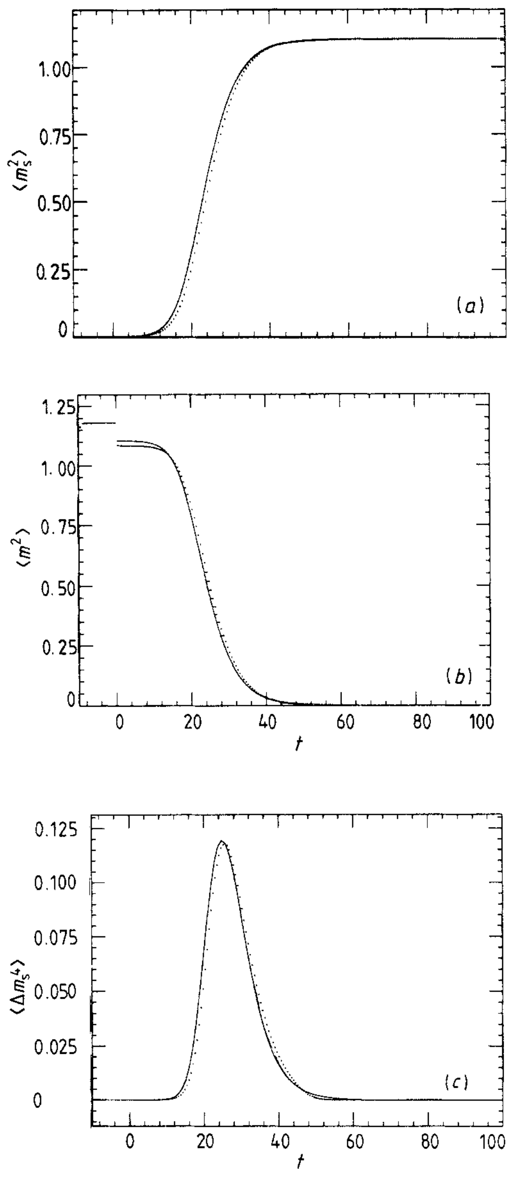

Figure 2. Comparison between the results of the QDT and computer simulation for the total and staggered average magnetisations $m(t)$ and $m_s(t)$, plotted in (a) and (b), respectively. In (c) the anomalous fluctuations (equation (33)) associated with the growth of m_s in the transient are shown. The quantities arising from simulations are obtained from the following definitions:

$$\langle m(t)^2 \rangle = \left\langle \frac{1}{2N} \left(\sum_i \varphi_i(t) + \psi_i(t) \right)^2 \right\rangle$$

$$\langle m_s(t)^2 \rangle = \left\langle \frac{1}{2N} \left(\sum_i \varphi_i(t) - \psi_i(t) \right)^2 \right\rangle$$

$$\langle \Delta m_s(t)^4 \rangle = \left\langle \frac{1}{2N} \left(\sum_i \varphi_i(t) - \psi_i(t) \right)^4 \right\rangle - \left\langle \left(\frac{1}{2N} \left(\sum_i \varphi_i(t) + \psi_i(t) \right)^2 \right)^2 \right\rangle .$$

The values of the parameters used in (a), (b) and (c) are

$$\varepsilon = 0.01 \qquad r_0 = 10 \qquad U = 10$$
$$N = 100 \qquad J_F = 1 \qquad J_{AF} = 0.1.$$

Here one can see that at the very instant the external field is switched off there is a very fast relaxation of the magnetisation.

process. Further, we introduce the time average of the process over finite times. This kind of time average will become, asymptotically, a deterministic quantity. If this asymptotic value does not vanish on a macroscopic time scale (of the order of magnitude of the size of the system), a spontaneous growth of order will occur. From a theoretical point of view, time-averaged quantities are well approximated by the QDT limit. We shall consider in the following the time average of the squared staggered magnetisation. A very detailed description of the ordering kinetics is given by the study of the probability distribution functions (PDFs) $P(m_s^2)$ and $P(\overline{m_s^2})$ associated, respectively, to the square of the staggered magnetisation and its time average $\overline{m_s^2}(t)$ defined by

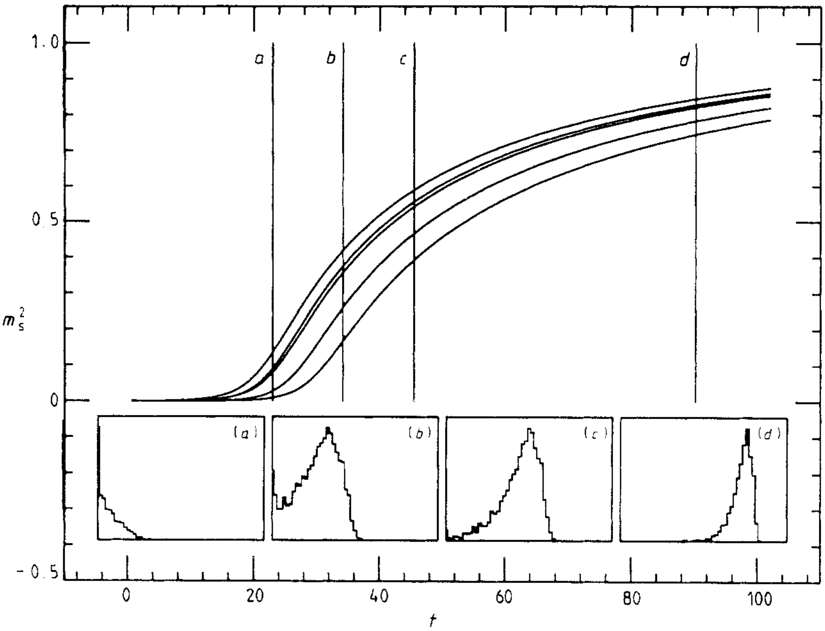

Figure 3. Sketch of the sampling procedure for obtaining the PDF of the time-averaged staggered magnetisation given in equation (24*a*). Histograms (*a*)–(*d*) for different realisations of the process are taken at the times marked by the vertical lines *a*–*d*.

$$\overline{m_s^2}(t) = \frac{1}{t}\int_0^t m_s^2(t')\,dt'. \tag{34}$$

In (34) the small fluctuations in the final regime, neglected in the QDT, are obscured by the average process while the transient fluctuations are still present. The PDF can be obtained numerically by sampling $m_s^2(t)$ or $\overline{m_s^2}(t)$ at different times for different thermal realisations of the process, as shown schematically in figure 3 for $P(\overline{m_s^2})$. Its analytic form can instead be deduced by considering the explicit expression of $\overline{m_s^2}(t)$ obtained from (34) and from the definition (10*b*) by means of equations (27)–(32):

$$\overline{m_s^2}(t) = \frac{1}{t}R_{eq}^2\int_0^t \frac{(1-\exp(-4J_{AF}t')y^2}{(1-y^2)\exp(-4J_{AF}t'+y^2)}\,dt' \tag{35}$$

where y is a zero-average Gaussian variable with variance σ given by (24*c*). After a straightforward integration one gets

$$\overline{m_s^2}(t) = R_{eq}^2[\![1+\{\ln[(1-y^2)\exp(-4J_{AF}t)+y^2]\}/4J_{AF}t(1-y^2)]\!]. \tag{36}$$

The generating function of the above variable is

$$G_{\overline{m_s^2}(t)}(\omega) = \frac{\exp(\omega R_{eq}^2)}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}\exp(-y^2/2\sigma^2)[(1-y^2)$$

$$\times\exp(-4J_{AF}t)+y^2]\frac{\omega R_{eq}^2}{4J_{AF}t(1-y^2)}\,dy.$$

In principle one can obtain all the moments $(\overline{m_s^2})^k$ from this generating function. However, in order to have a quantity that can be directly compared to numerical simulations we concentrate our attention on the probability $P(\overline{m_s^2})$. Defining $P(\overline{m_s^2})$ as the probability that $\overline{m_s^2}(t)$ has a given value $\overline{m_s^2}$ one can write

$$P(\overline{m_s^2}) = \frac{\mathrm{d}y}{\mathrm{d}\overline{m_s^2}} \exp(-y^2/2\sigma^2)\sqrt{2\pi}\sigma. \tag{37}$$

In (37) the inverse Jacobian of the transformation is

$$\frac{\mathrm{d}\overline{m_s^2}}{\mathrm{d}y} = \frac{R_{\mathrm{eq}}^2 y}{2J_{\mathrm{AF}}t(1-y^2)}\left(\frac{\ln[y^2+(1-y^2)\exp(-4J_{\mathrm{AF}}t)]}{(1-y^2)} + \frac{1-\exp(-4J_{\mathrm{AF}}t)}{(1-y^2)\exp(-4J_{\mathrm{AF}}t)+y^2}\right) \tag{38}$$

and y must be understood as a function of $\overline{m_s^2}$ obtained by inversion of equation (36). The probability (37) is singular for $y = 0$ but the moments of $\overline{m_s^2}(t)$ converge since for $y \to 0$:

$$(\overline{m_s^2})^k \simeq R_{\mathrm{eq}}^3\left(\frac{\exp(4J_{\mathrm{AF}}t)-1}{4J_{\mathrm{AF}}t} - 1\right)^k y^{2k}.$$

Hence one can evaluate $P(\overline{m_s^2})$ at any fixed time. The analytic results are compared to the histograms obtained by simulations in figure 3. The PDF $P(m_s^2)$ can be obtained in an analogous way.

In analogy with the previous case, and making use of equation (23), m_s can be obtained from the relation

$$\omega^2 = -z^2(t)m_s^2/(m_s^2 - R_{\mathrm{eq}}^2). \tag{39}$$

With the same procedure used for (38) one arrives at

$$P(m_s^2) = (1/\sqrt{2\pi})(\mathrm{d}\omega/\mathrm{d}m_s^2)\exp[-z^2(t)m_s^2/2(R_{\mathrm{eq}}^2 - m_s^2)]$$

where

$$\mathrm{d}\omega/\mathrm{d}m_s^2 = [R_{\mathrm{eq}}^2/2\sqrt{z^2(t)m_s^2/(R_{\mathrm{eq}}^2 - m_s^2)}][z(t)/(R_{\mathrm{eq}}^2 - m_s^2)]^2$$

and whose variance is just equation (33) for the anomalous fluctuations of m_s^2.

6. Conclusions

In the present paper we have presented a simple model for the relaxation of an antiferromagnetic spin system initially forced to be in a ferromagnetic state by means of an external field. Choosing the simplest spin arrangement of two interpenetrating sublattices and introducing the spherical model we were able to state the non-linear dynamical process in terms of coupled Langevin equations. We have investigated the problem essentially in two steps. In the first one we have reduced the $2N$ spin degrees of freedom to only two macroscopic degrees of freedom with the same statistical properties as the original set. Moreover, we have individuated the conditions for an adiabatic decay of one variable with respect to the other. In the second step the dynamical behaviour has been obtained by decoupling the two Langevin equations in the adiabatic limit of weak antiferromagnetic interaction after rewriting them in terms of the modulus and phase of a vector whose components are the two macroscopic degrees of freedom

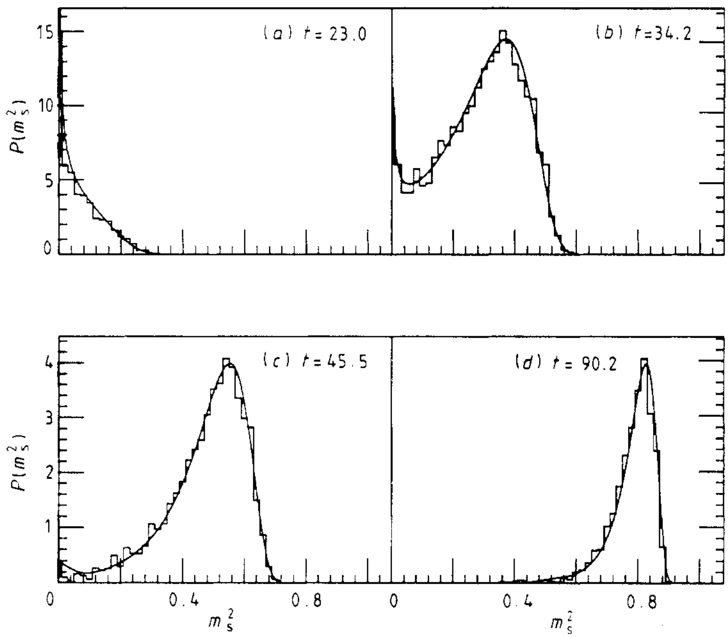

Figure 4. Probability distribution function of the time-averaged staggered magnetisation. The histograms are obtained by computer simulation (see figure 3) from a sample of 1200 processes averaged at the time, t, shown on each of plots (a)–(d). The continuous curve is the analytic QDT result, equation (37). At intermediate times, e.g. figure 3(b), the bimodal behaviour of the distribution is to be noted.

of the system, e.g. the total and staggered magnetisations. In this case the relaxation of these quantities occurs on a very different time scale. The modulus decays rapidly into a phase-independent equilibrium value, while the phase shows all the typical features of decay from an unstable state, already known in other descriptions (Suzuki 1980, de Pasquale and Tombesi 1979, de Pasquale *et al* 1982). As expected, due to the behaviour of the phase, the initial staggered magnetisation fluctuations are strongly enhanced to values larger than the steady state ones in the transient. This is a general feature of the decay from an unstable state (de Pasquale *et al* 1982). The problem shows close analogies with that of the growth of the $q = 0$ mode in a finite ferromagnetic system (Ciuchi *et al* 1988) where the magnetisation growth occurs because of the amplifications of initially small fluctuations due to the finite size effect. In fact the fluctuations were found to be of the order of $1/N$, as was the case in our study. Finally, we have compared theoretical and simulated probability distribution functions for the squared staggered magnetisation and for its time average, which is a more appropriate quantity to show the presence of anomalous fluctuations in the transient. (See for example figure 4 where five different realisations of the time-averaged staggered magnetisation process are shown.) The latter (time-averaged) quantity is also proposed as a possible measure of the spontaneous growth of order. The probability distribution shows a double peak structure at intermediate times and then shrinks around the steady state value (see figure 3). This bimodal shape is again a feature of the decay from an unstable state (Baras *et al* 1983). A very good agreement between theoretical results and numerical simulations (see figures 3, 4) is found.

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